MA125 Exam 1 Version 1

Name: KEY

1) A function f is defined piecewise by:

$$f(x) = \begin{cases} 2 & x \leq -1 \\ a^{-x} & -1 < x < 0 \\ 1 + \ln(x+b) & x \geq 0 \end{cases}$$

For what values (if any) of a and b is f continuous for all real numbers?

We need $\lim_{x \to -1^+} a^{-x} = f(-1) = 2 \Rightarrow a^{-(-1)} = 2 \Rightarrow a^1 = 2 \therefore a = 2$

Regardless of what $a > 0$ is, $f(0) = a^0 = 1$. a^{-x} is continuous everywhere, so $\lim_{x \to 0^-} a^{-x} = a^0 = 1$. This must equal $f(0) = 1 + \ln b$,

so $\ln b = 0 \Rightarrow b = 1$

2) Use the intermediate value theorem to prove that the equation

$$\ln(2 - x^2) = x$$

has at least one real root. You should include a justification that the conditions under which the theorem applies are satisfied.

Let $f(x) = \ln(2-x^2) - x$. Then f is continuous on $[-1,1]$,

and: $f(1) = \ln(2-1^2) - 1 = -1$
$f(-1) = \ln(2-(-1)^2) + 1 = +1$ (many other choices are possible)

By the IVT, since f is continuous on the closed interval $[-1,1]$, f assumes every value between $f(1) = -1$ and $f(-1) = 1$ somewhere in the interval $(-1,1)$. So there is a $c \in (-1,1)$ for which $f(c) = \ln(2-c^2) - c = 0 \Rightarrow$ The equation has a real root.

3) State the precise definition of a limit and use it to give a proof that

$$\lim_{x \to 4} 5x - 1 = 19$$

$\lim_{x \to a} f(x) = L$ if, for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

Let $\epsilon > 0$ be given. $5 \cdot 4 - 1 = 19$

$|f(x) - L| = |5x - 1 - 19| = |5x - 20| < \epsilon \iff -\epsilon < 5x - 20 < \epsilon$

$\iff -\frac{\epsilon}{5} < x - 4 < \frac{\epsilon}{5} \iff |x - 4| < \frac{\epsilon}{5} \Leftarrow$ choose $\delta = \epsilon/5$

4) Let

$$f(x) = 3e^{2x} - 1$$

Find the set of real numbers on which the **INVERSE** of f is continuous.

$f(x) = 3e^{2x} - 1 = y \qquad \frac{y+1}{3} = e^{2x} \qquad \ln\left(\frac{y+1}{3}\right) = 2x$

$x = \frac{1}{2} \ln\left(\frac{y+1}{3}\right)$ inverse: $y = \frac{1}{2} \ln\left(\frac{x+1}{3}\right)$

continuous if $\left(\frac{x+1}{3}\right) > 0 \Rightarrow x + 1 > 0 \Rightarrow x > -1$

5) A function is defined piecewise by:

$$\begin{cases} k & \text{if } x = 4 \\ f(x) & \text{if } x \neq 4 \end{cases}$$

If f has the property that

5 16 − 20 + 4

$$\frac{x^2 - 6x + 5}{1 - \sqrt{x}} \le f(x) \le \frac{x^2 - 5x + 4}{x - 4} \qquad x \in [-5, 5],$$

determine the value of k that makes f continuous at $x = 4$. You should explain what theorem(s) you used and how you applied them.

by direct subs,

$$\lim_{x \to 4} \frac{x^2 - 6x + 5}{1 - \sqrt{4}} = \frac{16 - 24 + 5}{1 - 2} = \frac{-3}{-1} = 3$$

$$x^2 - 5x + 4 = \frac{(x-4)(x-1)}{(x-4)} = x - 1 \text{ if } x \neq 4 \text{ limit} = 3$$

Let $k = 3$

6) Determine whether the following limit exists. If it exists, find its value.

$$\lim_{x \to -1^+} \frac{2x^2 - x - 3}{3x^2 + 4x + 1} = \frac{(x+1)(2x-3)}{(x+1)(3x+1)}$$

$$\begin{array}{r} 2x - 3 \\ x+1 \overline{) \, 2x^2 - x - 3} \\ 2x^2 + 2x \\ \hline -3x - 3 \end{array}$$

$$\begin{array}{r} 3x + 1 \\ x+1 \overline{) \, 3x^2 + 4x + 1} \\ 3x^2 + 3x \\ \hline x + 1 \end{array}$$

by direct subs, if $x \neq -1$,

$$f(x) = \left(\frac{2x - 3}{3x + 1}\right) = \frac{-2 - 3}{-3 + 1} = \frac{-5}{-2} = \frac{5}{2}$$

7
The position at time t of a particle moving along a straight line is given by

$$f(t) = (t + a)^2$$

where a is a constant chosen to make the particle's **average velocity** from $t = 0$ to $t = 2$ equal to 4.

a) What is the value of a?

b) What is the instantaneous velocity at $t = 1$?

$f(t) = t^2 + 2at + a^2$

$V_{avg} = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{4 + 4a + a^2 - a^2}{}$

$4 + 4a = 4$ $\quad$ $4 + 4a = 8$

$4a = 0$ $\quad$ $4a = 4$

$a = 0$ $\quad$ $a = 1$

$f(t) = t^2$ $\quad$ $f(t) = (t+1)^2 = t^2 + 2t + 1$

~~f~~ ~~2t~~

inst: $\lim_{h \to 0} \dfrac{(t+h)^2 + 2(t+h) + 1 - (t^2 + 2t + 1)}{h}$

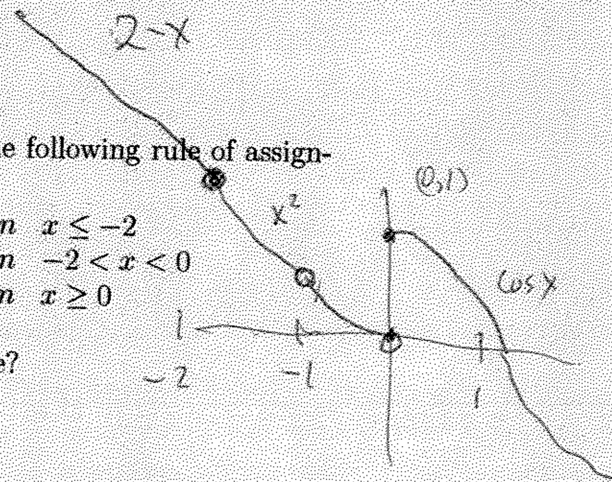
$= \dfrac{\cancel{t^2} + 2th + h^2 + \cancel{2t} + 2h + \cancel{1} - \cancel{t^2} - \cancel{2t} - \cancel{1}}{h}$

$= \dfrac{2th + h^2 + 2h}{h} = 2t + 2 = 4$

(OVER)

8 A function $f(x)$ is defined piecewise by the following rule of assignment:

$$f(x) = \begin{cases} 2 - x & \textit{when} \quad x \leq -2 \\ (x^4 - x^2)/(x^2 - 1) & \textit{when} \quad -2 < x < 0 \\ \cos x & \textit{when} \quad x \geq 0 \end{cases}$$



Which of the following statements are true?

(T) $\lim_{x \to -2^-} f(x)$ exists

(T) $\lim_{x \to -2^+} f(x)$ exists

(T) $\lim_{x \to -2} f(x)$ exists

(T) $f(x)$ is continuous from the left at $x = -2$

(T) $f(x)$ is continuous from the right at $x = -2$

(T) $f(x)$ is continuous at $x = -2$

(T) $\lim_{x \to -1^-} f(x)$ exists

(T) $\lim_{x \to -1^+} f(x)$ exists

(T) $\lim_{x \to -1} f(x)$ exists

(F) $f(x)$ is continuous at $x = -1$

(F) $f(x)$ is continuous from the left at $x = 0$

(T) $f(x)$ is continuous from the right at $x = 0$

(T) $f(x)$ is continuous everywhere except -1 and 0

(T) $f(x)$ is continuous at $x = \pi/2$

(T) $f(x)$ is continuous on the interval $[\pi/2, \infty)$

(T) $f(x)$ is continuous on the interval $(-\infty, -1.5]$

$$\frac{x^2(x^2-1)}{(x^2-1)}$$

$f(x) = x^2$

if $(x^2-1) \neq 0$

$x \neq \pm 1$

[illegible]

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