MA125 Exam 2

Name:

1) Given two functions f and g and a real number a such that

$$f(a) = 0 \quad f'(a) = 1 \quad g(a) = 2$$

find the **y-intercept** of the line tangent to the quotient function

$$y = \left(\frac{f}{g}\right)(x) \qquad \text{at } x = a$$

$$y'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2} = \frac{2 \cdot 1 - 0 \cdot g'(a)}{2^2} = \frac{2 \cdot 1}{2^2} = \frac{1}{2}$$

Slope Intercept

$y - \left(\frac{f}{g}\right)(a) = y'(a)(x-a)$

$y - \frac{0}{2} = \frac{1}{2}(x-a)$ $\quad y = \frac{1}{2}x - \frac{a}{2}$

answer: $-\frac{a}{2}$

2 An object is launched from the ground at an angle. The vertical position (y-coordinate) and horizontal position (x-coordinate) after t seconds are given by the functions:

$\frac{dx}{dt} = 50$ ← horizontal velocity

$$x(t) = 50t \quad \text{and} \quad y(t) = 160t - 16t^2$$

$\frac{dy}{dt} = 160 - 32t$

vertical velocity

$\frac{d^2y}{dt^2} = -32$

Vertical accel.

a) What is the *horizontal* velocity at $t = 1$? 50

b) What is the *vertical* acceleration at $t = 3$? -32

c) At what time(s), if any, is the *vertical* velocity zero? $160 - 32t = 0 \Rightarrow t = \frac{160}{32} = 5$

d) At the instant the maximum y-coordinate is reached, what is the x-coordinate? max at $t=5$, $x = 50t = 250$

e) What is the **average** *vertical* velocity from $t = 0$ to $t = 10$?

$$V_{avg} = \frac{y(10) - y(0)}{10 - 0} = \frac{1600 - 16(100)}{10} = 0$$

3 The volume and radius of a sphere are related by the formula:

$$V = \frac{4}{3}\pi r^3$$

Air is being added to a spherical balloon in such a way that the compression is negligible.

a) What is the rate of change of the *radius* with respect to the *volume* when the balloon contains 1000 cm^3 of air?

b) What is the average rate of change of the radius as the volume changes from 100 to 400 cm^3?

a) $V = \frac{4}{3}\pi r^3 \quad r = r(V) = \sqrt[3]{\frac{3V}{4\pi}} = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$\frac{dr}{dV} = \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-2/3}$

b) $\Delta r_{Avg} = \frac{r(400) - r(100)}{400-100} = \frac{\sqrt[3]{\frac{1200}{4\pi}} - \sqrt[3]{\frac{300}{4\pi}}}{300}$

4 Find the horizontal and vertical asymptotes (if there are any) of the graph of the **derivative** $h'(x)$ of the function

$$h(x) = \frac{x-1}{x^2-1}$$

Where is $h'(x)$ continuous?

$h'(x) = \frac{(x^2-1)\cdot 1 - (x-1)(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2+2x}{(x^2-1)^2} = \frac{-(x^2-2x+1)}{(x^2-1)^2} = \frac{-(x-1)^2}{(x^2-1)^2}$

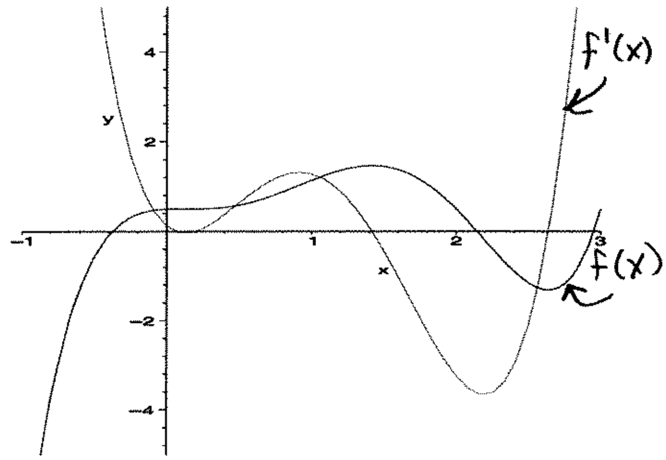
$h'(x) = \frac{-(x-1)^2}{[(x-1)(x+1)]^2} = \frac{-(x-1)^2}{(x-1)^2(x+1)^2}$

if $x \neq 1$, This reduces to: $-\frac{1}{(x+1)^2}$

Vertical asymptote at $x = -1$

A rational function is continuous on its domain, which is everywhere except the real roots of the denominator, ± 1 in this case.

5 Given the following graphs of $f(x)$ and $f'(x)$,



a) Identify which curve is $f(x)$ and which is $f'(x)$.

See diagram

b) Identify the intervals on which $f(x)$ is increasing

f is increasing where $f'(x) > 0 \Rightarrow (-1, 0.2) \cup (0.2, 1.4) \cup (2.6, 3)$ (approximately)

c) Identify the intervals on which $f(x)$ is concave down

f(x) is concave down where f'(x) is decreasing $(-1, 0.2) \cup (0.9, 2.1)$

d) Identify points at which the **second derivative** $f''(x)$ is zero.

$f''(x) = 0$ where $f'(x)$ has a horizontal tangent: $x = 0.2,\ 0.9,\ 2.1$ (approx)

e) Identify the intervals on which the **second derivative** $f''(x)$ is positive.

$f''(x)$ is positive where $f'(x)$ is increasing:

$(0.2, 0.9) \cup (2.1, 3)$ approximately

6 If $f(x) = x^3 - 3x + 2$ and $g(x) = e^x + 1$, find the derivative of the function defined by:

$$\left(\frac{f}{g}\right)(x)$$

On what interval is the derivative continuous?

$$\left(\frac{f}{g}\right)'(x) = \frac{(e^x+1)(3x^2-3) - (x^3-3x+2)e^x}{(e^x+1)^2} = 3e^x x^2 - 5e^x + 3x^2 - 3 - e^x x^3 + 3xe^x$$

$\left(\frac{f}{g}\right)'(x)$ is continuous everywhere since $e^x > 0$ for any $x \Rightarrow e^x + 1 \neq 0$

7 Find the derivative of the following function **directly from the definition of the derivative as a limit** (a is a constant):

$$f(x) = \frac{1}{\sqrt{x+a}}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h+a}} - \frac{1}{\sqrt{x+a}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x+a} - \sqrt{x+h+a}}{\sqrt{x+h+a}\sqrt{x+a}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(\sqrt{x+a} - \sqrt{x+h+a})}{\sqrt{x+h+a}\cdot\sqrt{x+a}}\left(\frac{\sqrt{x+a}+\sqrt{x+h+a}}{\sqrt{x+a}+\sqrt{x+h+a}}\right)}{h} = \lim_{h\to 0} \frac{\frac{\cancel{x}+\cancel{a}-\cancel{x}-\cancel{h}-\cancel{a}}{\sqrt{x+h+a}\cdot\sqrt{x+a}\left(\sqrt{x+a}+\sqrt{x+h+a}\right)}}{\cancel{h}}$$

$$= \frac{1}{2\sqrt{x+a}^{3/2}}$$

8 A function is defined piecewise by

$$f(x) = \begin{cases} x^3 + \sqrt[3]{x^4} + 3bx + c & x < 0 \\ 2x^4 - 5 + e^x & x \geq 0 \end{cases}$$

a) What value of c makes f continuous at $x = 0$?
b) What values of b and c make f continuous and differentiable at $x = 0$?

a) $f(0) = -5 + e^0 = -4 \Rightarrow \lim_{x\to 0^-} x^3 + \sqrt[3]{x^4} + 3bx + c$ must be -4

By direct substitution, The Limit is c so $c = -4$

b) $f'(0) = \begin{cases} 3x^2 + \frac{4}{3}\sqrt[3]{x} + 3b \\ 8x^3 + e^x & x \geq 0 \end{cases}$

$f'(0) = 1 \Rightarrow \lim 3x^2 + \frac{4}{3}\sqrt[3]{x} + 3b$ must be $1 \Rightarrow b = \frac{1}{2}$

9 Newton's law of gravitation states that the attractive force exerted on a body of mass m by a body of mass M is

$$F = \frac{GMm}{r^2}$$

where G is a constant and r is the distance between them in kilometers.

What is the rate of change of the force with respect to r when the distance between the objects is $10km$?

What is the average rate of change of the force with respect to r when the distance between the objects increases from $20km$ to $50km$?

$F(r) = \frac{GMm}{r^2}$ $\quad \frac{dF}{dr} = (-2)\frac{GMm}{r^3}$ $\quad \left.\frac{dF}{dr}\right|_{r=10} = \frac{-2GMm}{1000}$

$\Delta F_{Avg} = \frac{F(50)-F(20)}{30} = \frac{GMm}{30}\left(\frac{1}{50^2} - \frac{1}{30^2}\right)$

10 Find the equation of the line tangent to the curve

$$f(x) = e^x\left(\frac{x^2+2}{\sqrt{x}}\right)$$

at $x = 1$.

first Simplify!

$f(x) = e^x \cdot x^{3/2} + 2e^x x^{-1/2}$

Product Rule:

$f'(x) = e^x \cdot x^{3/2} + \frac{3}{2}x^{1/2}e^x + 2e^x x^{-\frac{1}{2}} - e^x x^{-3/2}$

$f'(1) = e^1 + \frac{3}{2}e^1 + 2e^1 - e^1 = \frac{7e}{2}$

$f(1) = e^1\left(\frac{1+2}{1}\right) = 3e$

$y - f(1) = f'(1)(x-1)$

$y - 3e = \frac{7e}{2}(x-1)$

or $\quad y = \frac{7e}{2}x - \frac{e}{2}$